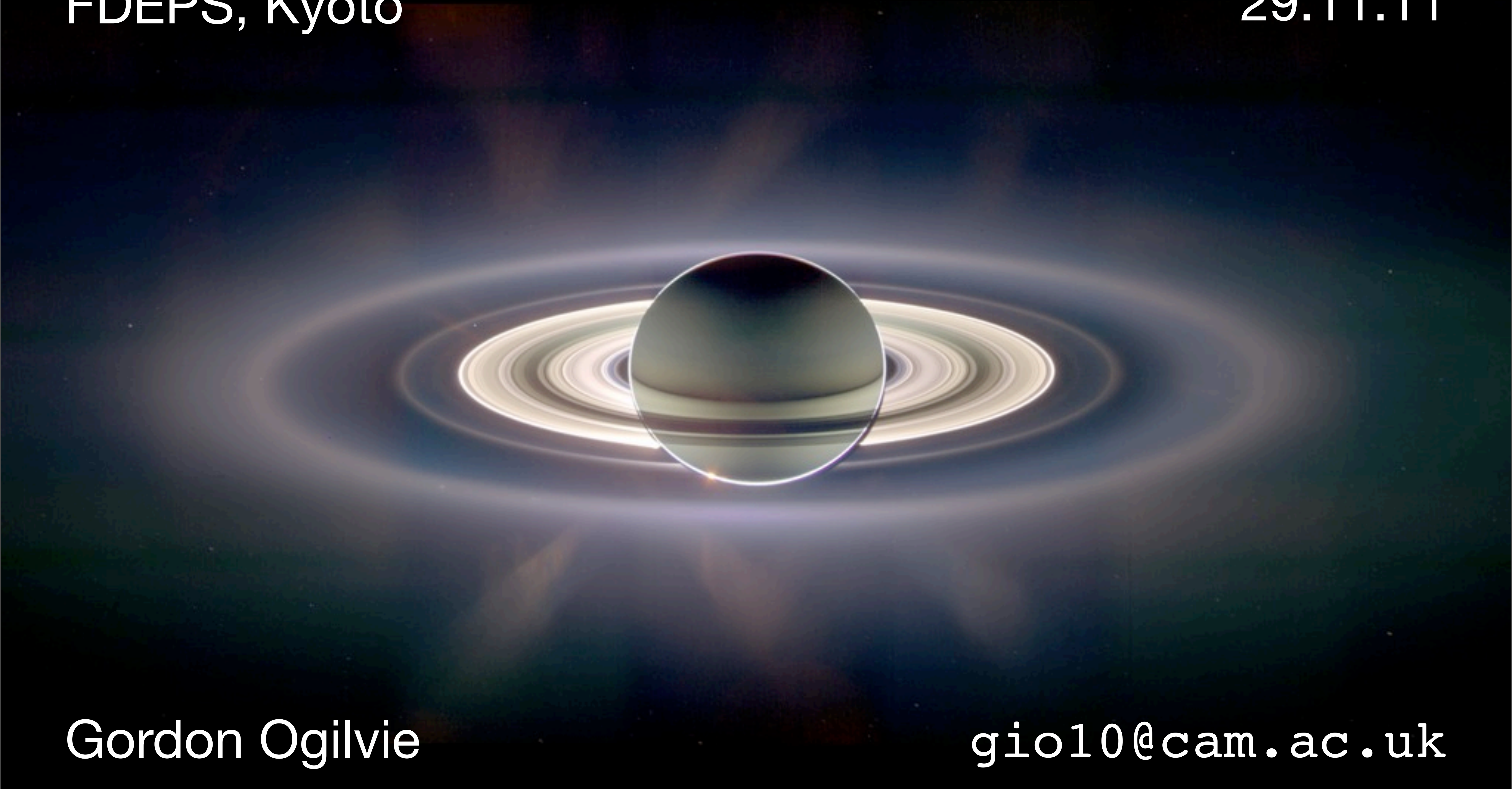
Dynamics of astrophysical discs

Lecture 1

FDEPS, Kyoto

29.11.11



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NASA

Outline

Lecture 1: Introduction to astrophysical discs

- Occurrence of discs, physical and observational properties
- Orbital dynamics, mechanics of accretion
- Equations of astrophysical fluid dynamics and MHD

Lecture 2: Evolution and structure of discs

- Evolution of an accretion disc
- Vertical disc structure, timescales

Lecture 3: Local approximation and incompressible dynamics

- Shearing sheet, shearing waves
- Incompressible dynamics: hydrodynamic stability, vortices

Outline

Lecture 4: Compressible dynamics of astrophysical discs

- Compressible dynamics: density waves, gravitational instability
- Satellite-disc interaction

Lecture 5: Magnetohydrodynamics of astrophysical discs

- Magnetorotational instability
- Jet launching

Seminar: Astrophysical tides and planet–star interactions

Astrophysical discs

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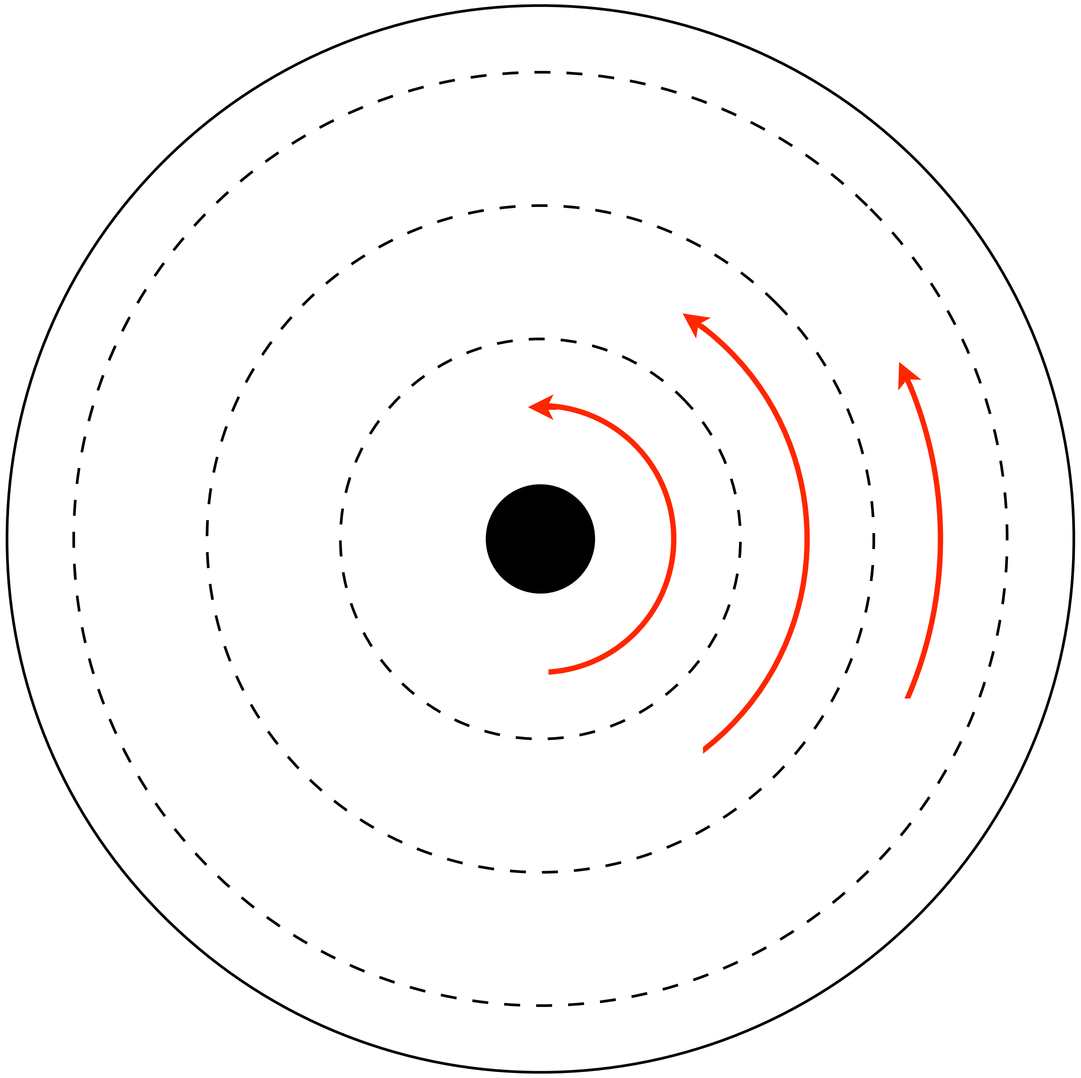
Continuous medium in orbital motion around a massive central body

orbital dynamics /
celestial mechanics ⇔ fluid dynamics /
continuum mechanics



- Usually circular, coplanar and thin
- Usually Keplerian (dominated by gravity of central mass)

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

- Shearing, dissipative systems

Astrophysical discs

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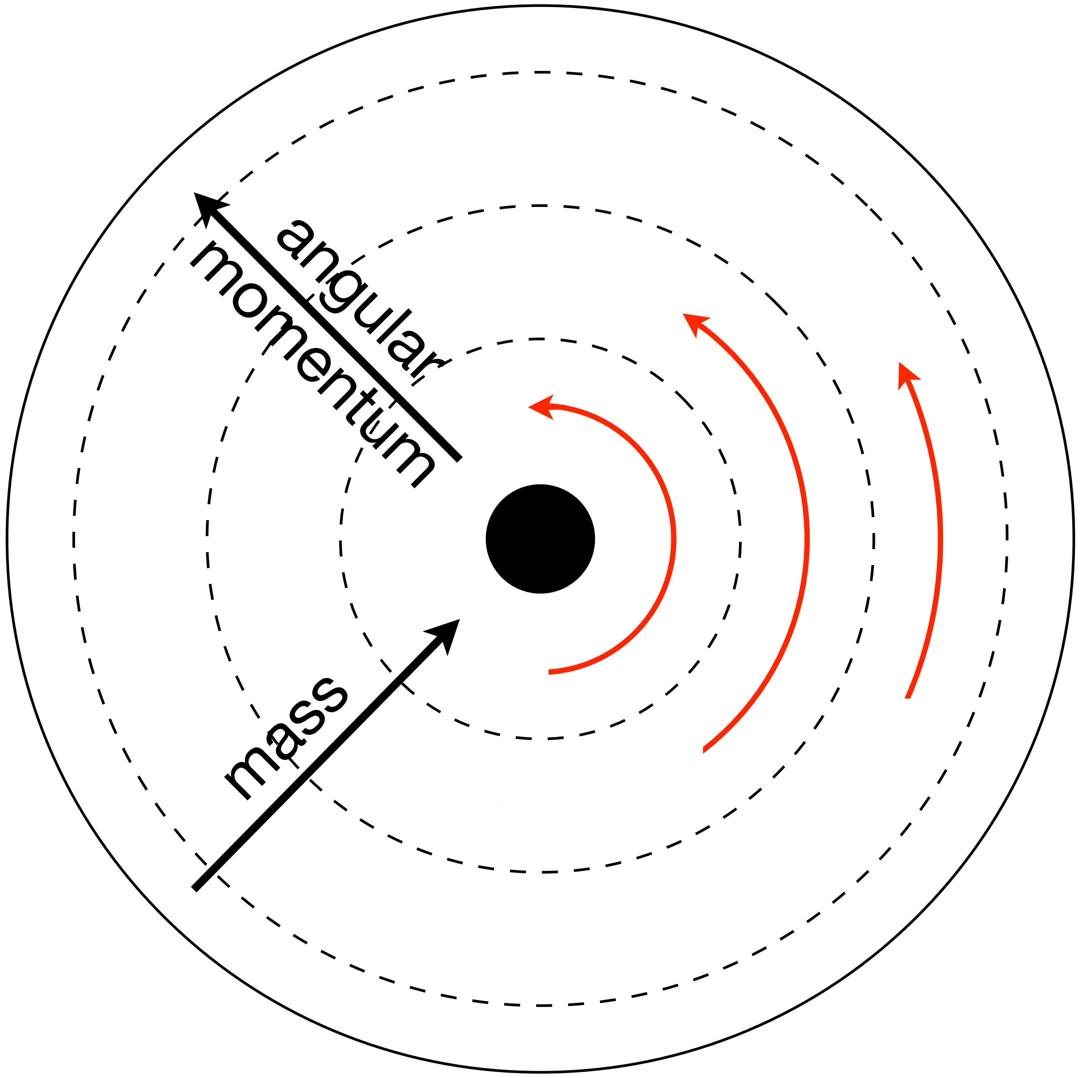
Continuous medium in orbital motion around a massive central body

orbital dynamics / celestial mechanics ⇔ fluid dynamics / continuum mechanics



- Usually circular, coplanar and thin
- Usually Keplerian (dominated by gravity of central mass)

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

- Shearing, dissipative systems
- Accretion disc (angular momentum out, mass in, energy liberated)

Occurrence of discs

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- Spiral galaxies (different: dark matter, stars, time-scales)
- Active galactic nuclei, quasars
- Interacting binary stars
- Protostellar / protoplanetary discs, solar nebula
- Planetary rings, circumplanetary discs
- Very rapidly rotating stars (Be stars)
- Exotica : supernovae, gamma-ray bursts, ...

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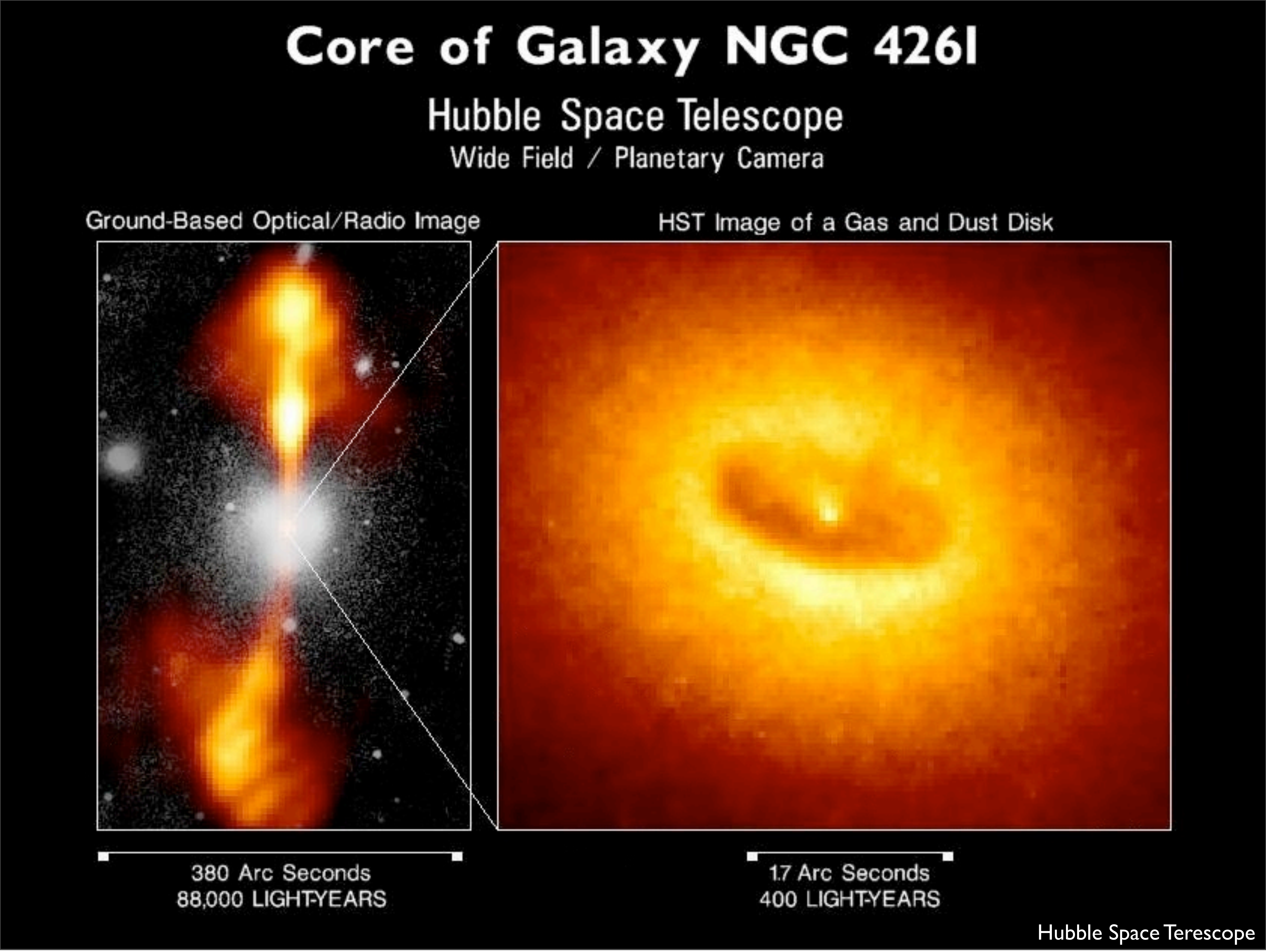
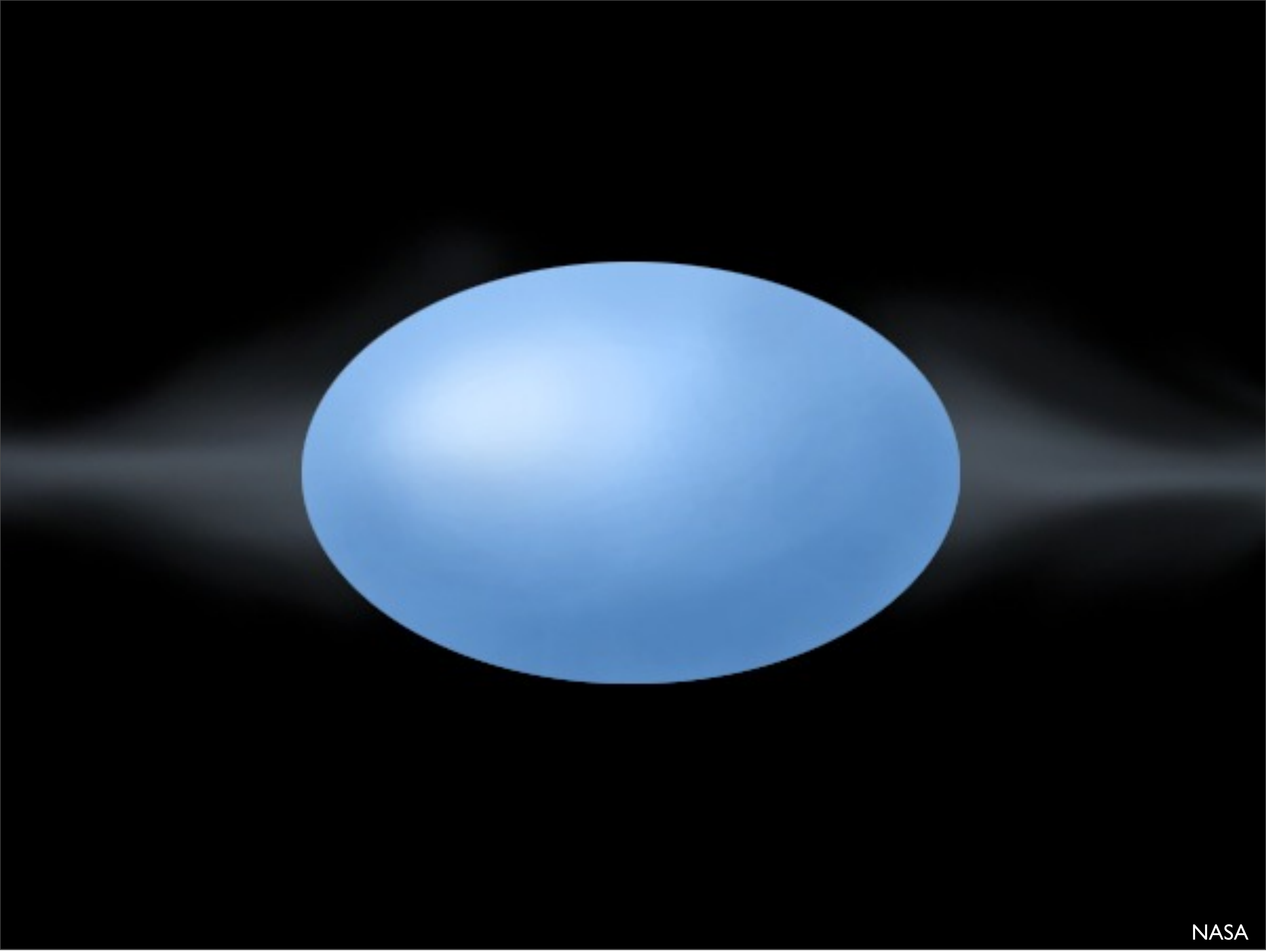


Hubble Space Terescope



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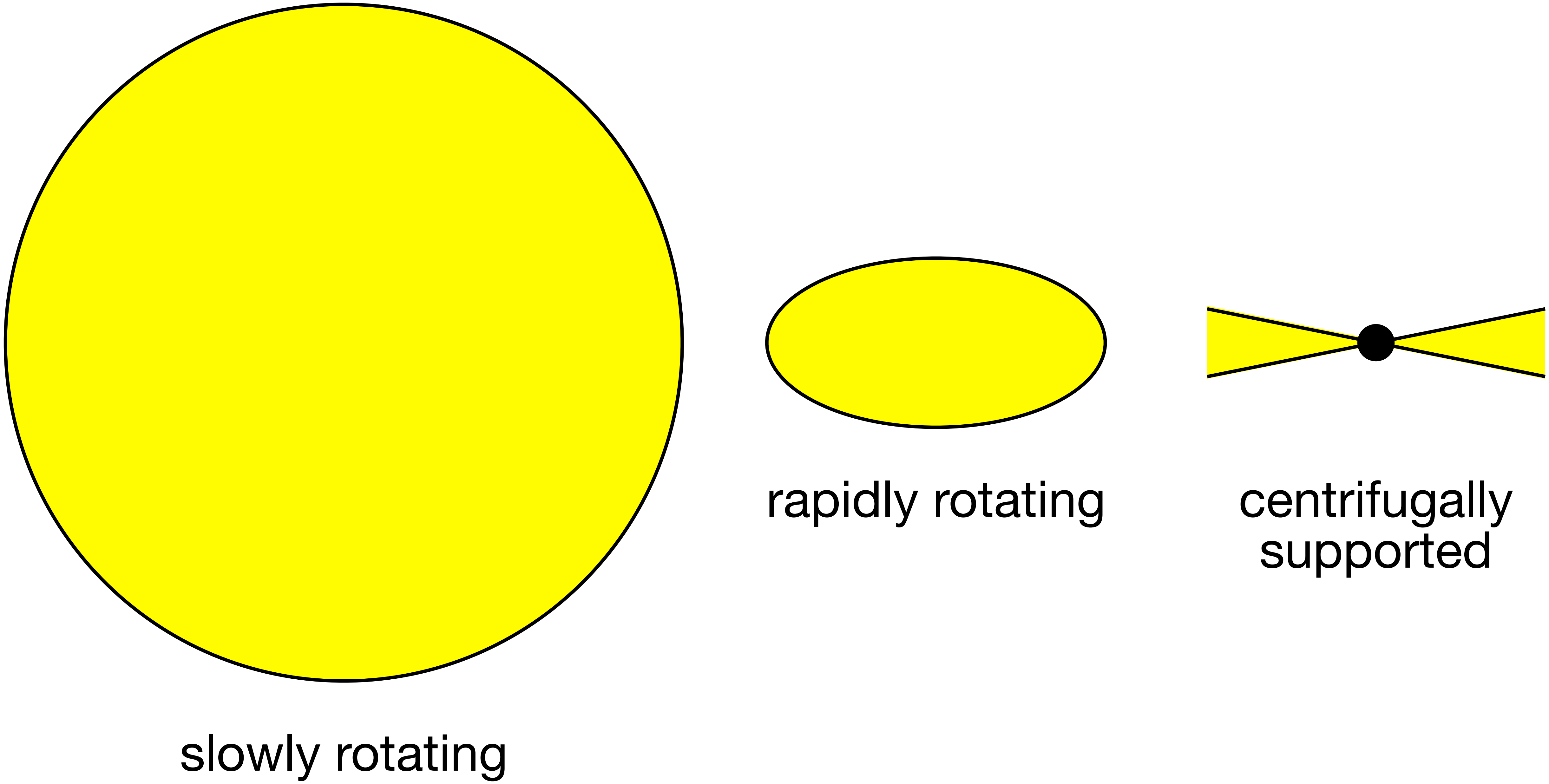
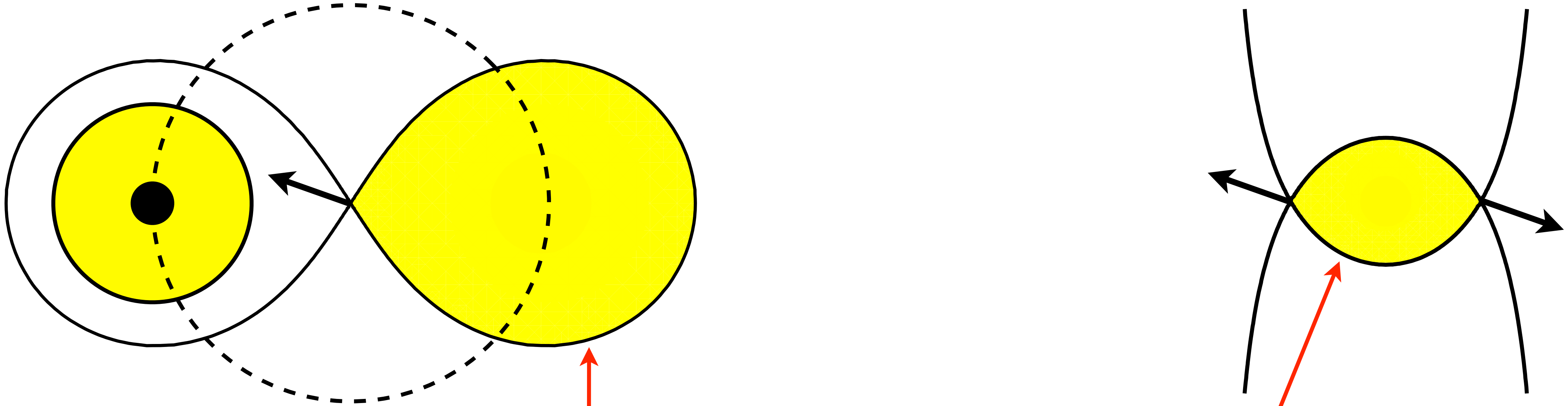
Formation of discs

- Collapse of rotating cloud (e.g. star formation)



Formation of discs

- Mass transfer / tidal disruption / merger



binary star

satellite

critical equipotential
(Roche lobe)

Hill “sphere”

- Other scenarios: captured stellar winds, stellar pulsations, ...

disc or *disk* ?

Physical composition

- Weakly ionized H / H_2 gas + solid particles (protoplanetary discs)
  aspect ratio $H/R \lesssim 0.1$, temperature $10\,\mathrm{K} \lesssim T \lesssim 10^3\,\mathrm{K}$
- Dense H / He plasma (interacting binary stars, AGN)
  aspect ratio $H/R \lesssim 0.03$, temperature $10^3\,\mathrm{K} \lesssim T \lesssim 10^7\,\mathrm{K}$
- Nuclear matter (exotica)
- Metre-sized iceballs (dense planetary rings)
  aspect ratio $H/R \sim 10^{-7}$, random velocity $\sim \mathrm{mm\,s^{-1}}$
- Dilute plasma (some cases of black-hole accretion flows)

Relevant descriptions:

- Gas dynamics
- Magnetohydrodynamics
- Kinetic theory

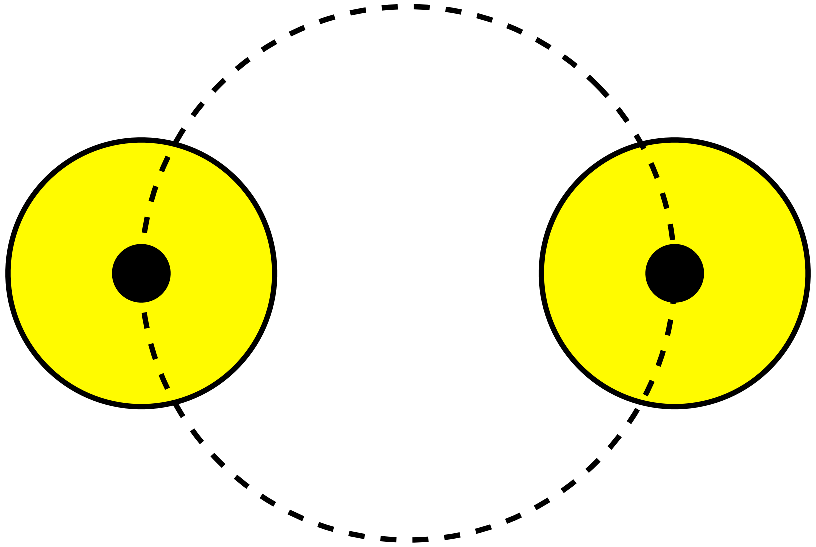
[+ relativity
+ radiation forces
where needed]

Characteristic length-scales and orbital periods

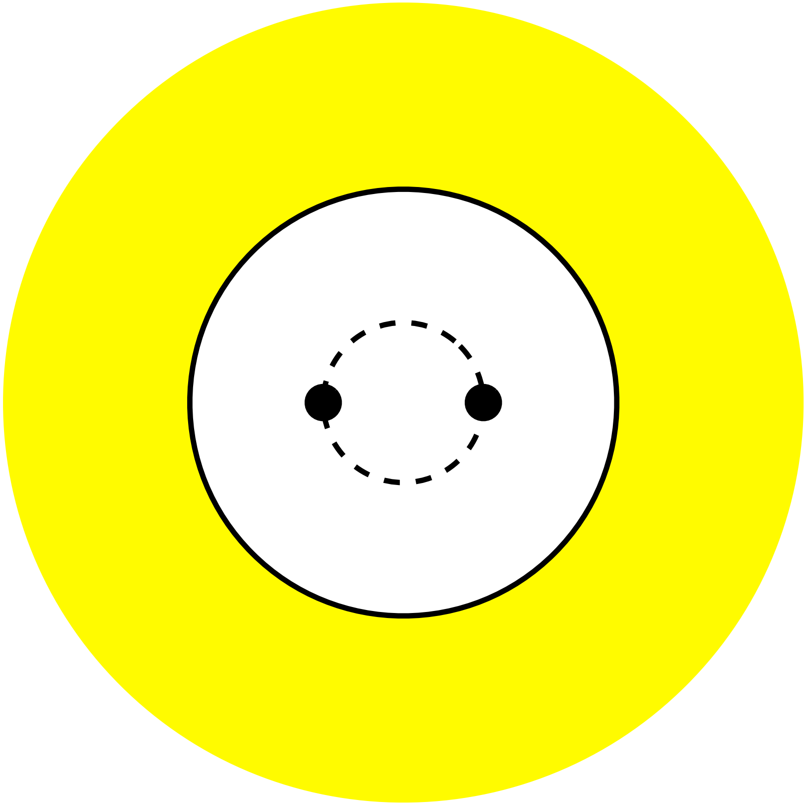
- Planetary ring: $r \sim 10^5\,\mathrm{km},\ t \sim 10\,\mathrm{hr}$
- Protoplanetary disc: $r_\mathrm{out} \sim 100\,\mathrm{AU},\ t_\mathrm{out} \sim 1000\,\mathrm{yr}$
  $r_\mathrm{in} \sim 0.01\,\mathrm{AU},\ t_\mathrm{in} \sim 10\,\mathrm{day}$
- X-ray binary star: $r_\mathrm{out} \sim R_\odot,\ t_\mathrm{out} \sim \mathrm{hr} - \mathrm{day}$
  $r_\mathrm{in} \sim 10\,\mathrm{km},\ t_\mathrm{in} \sim 10^{-3}\,\mathrm{s}$
- AGN: $r_\mathrm{out} \sim 0.1\,\mathrm{pc},\ t_\mathrm{out} \sim 1000\,\mathrm{yr}$
  $r_\mathrm{in} \sim \mathrm{AU},\ t_\mathrm{in} \sim \mathrm{hr}$

- Parsec $\mathrm{pc} = 3.086 \times 10^{18}\,\mathrm{cm}$
- Astronomical unit $\mathrm{AU} = 1.496 \times 10^{13}\,\mathrm{cm}$
- Solar radius $R_\odot = 6.960 \times 10^{10}\,\mathrm{cm}$

Configurations: binary stars

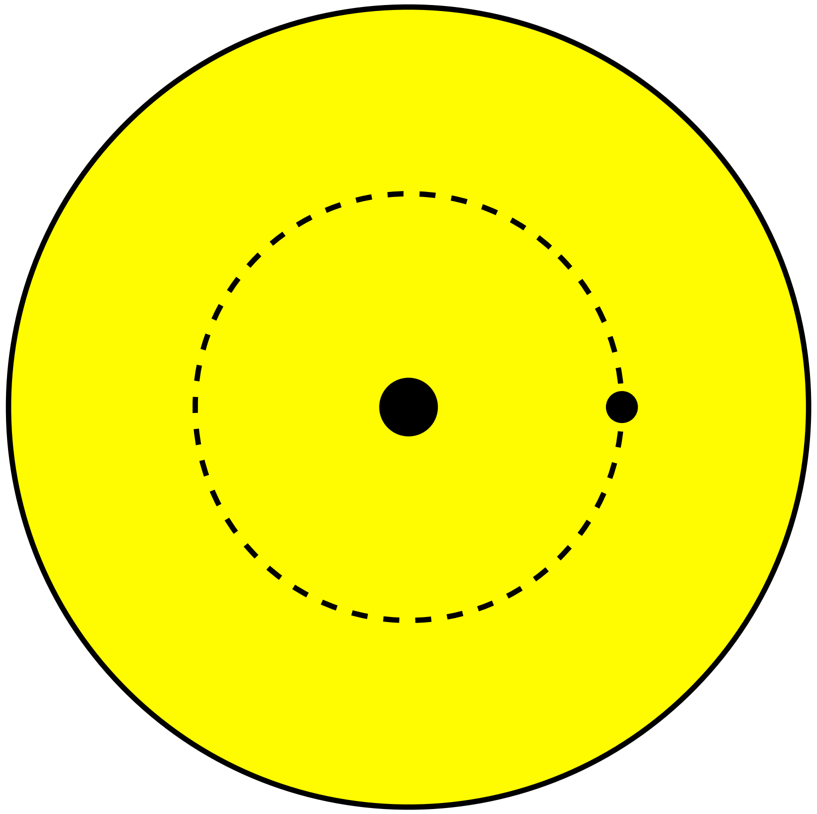


circumstellar disc(s)

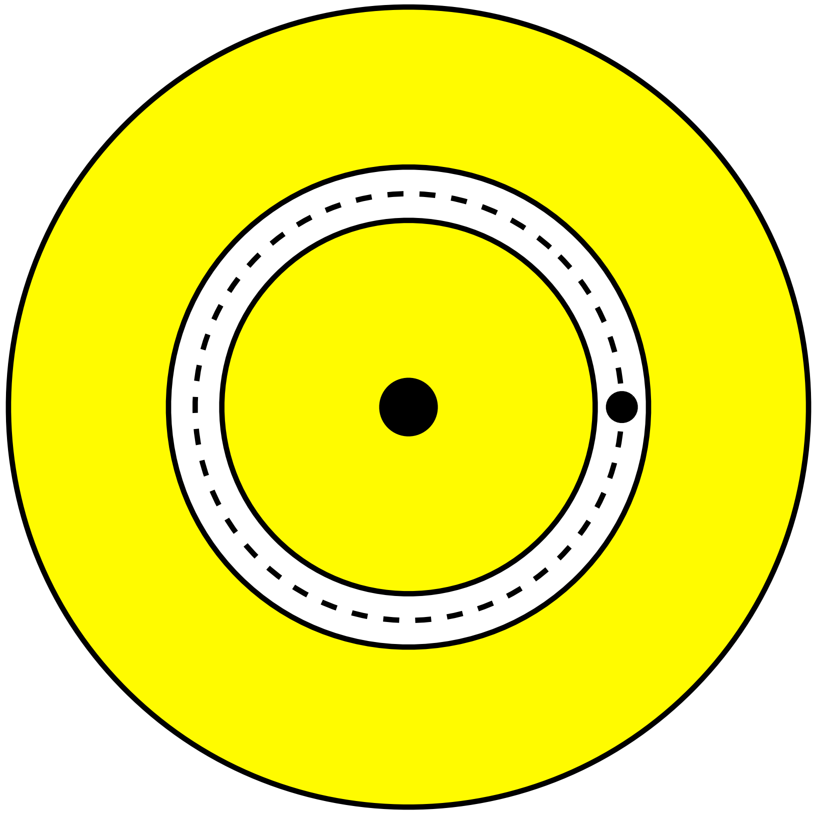


circumbinary disc

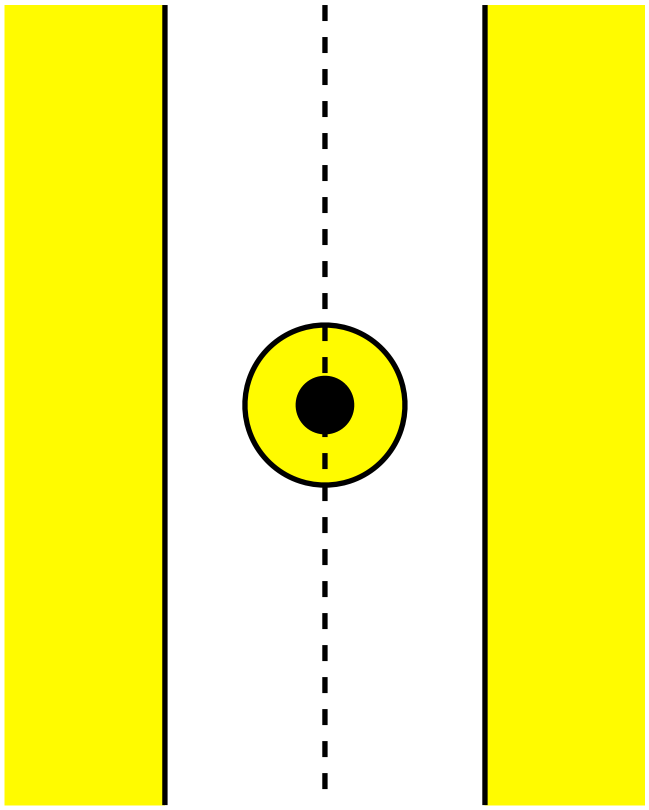
Configurations: protoplanetary systems



embedded planet



gap-opening planet
interior + exterior discs



circumplanetary disc

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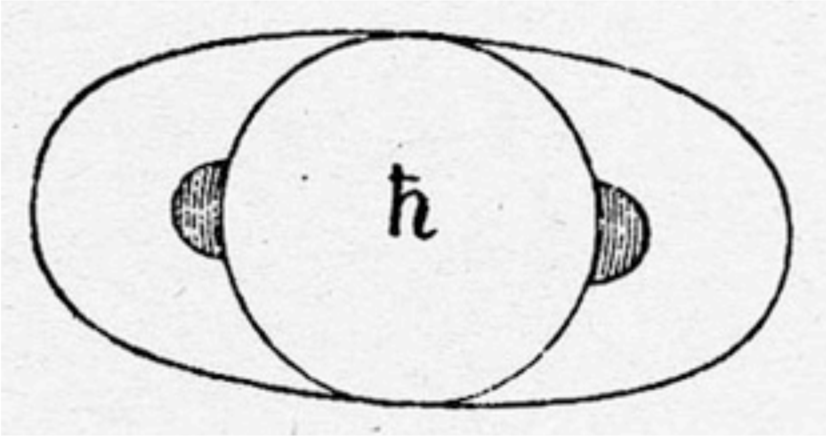
Observations: Saturn’s rings

- Galileo (1610)

SMAISMRMILMEPOETALEUMIBUNENUGTTAUIRAS

ALTISSIMUM PLANETAM TERGEMINUM OBSERVAVI

I have observed the most distant planet to have a triple form



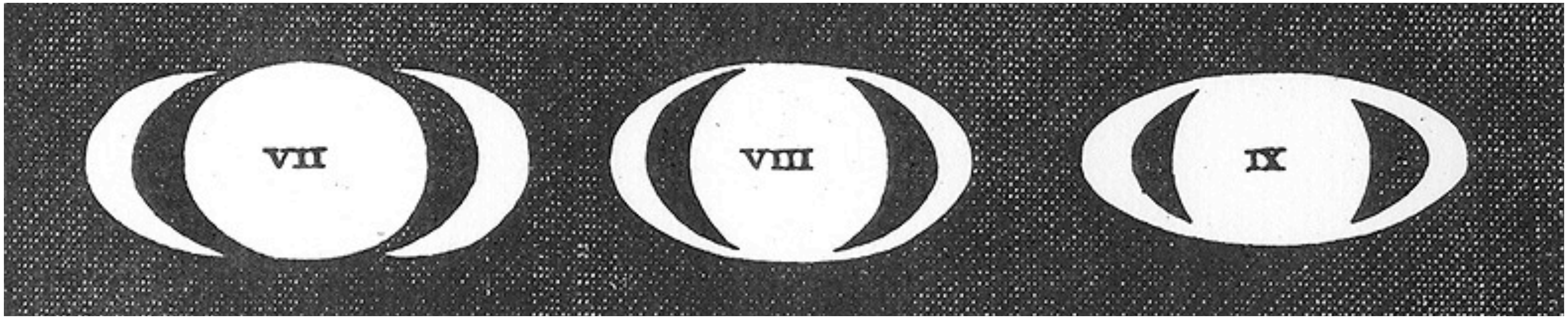
Galileo, 1610

- Huygens (1659)

AAAAAACCCCCDEEEEEHIIIIIIILLLLMMNNNNNNNNNOOOOPPQRRSTTTTTUUUUU

ANNULO CINGITUR TENUI PLANO NUSQUAM COHAERENTE AD ECLIPTICAM INCLINATO

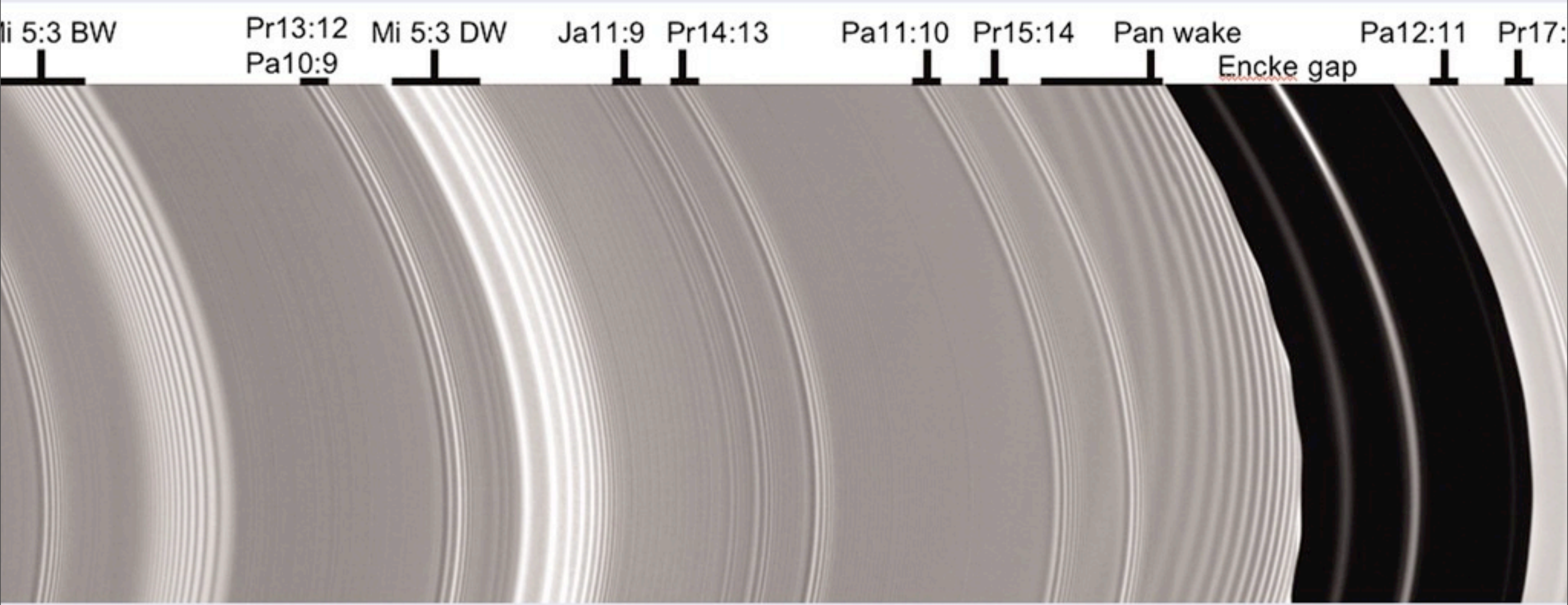
It is surrounded by a thin flat ring, nowhere touching, and inclined to the ecliptic



Huygens,1659:System saturnium, pp.84

- Hooke, Cassini, ..., Laplace, Maxwell, ...
- *Voyager 1* and 2 flybys (1980-1)
- *Cassini* in orbit (2004-)

Observations: Saturn’s rings



- waves, wakes, gaps, ringlets, braids, shepherds, propellers, ...
- `ciclops.org`

Cuzzi, et al., 2010:Science, 327, 1470

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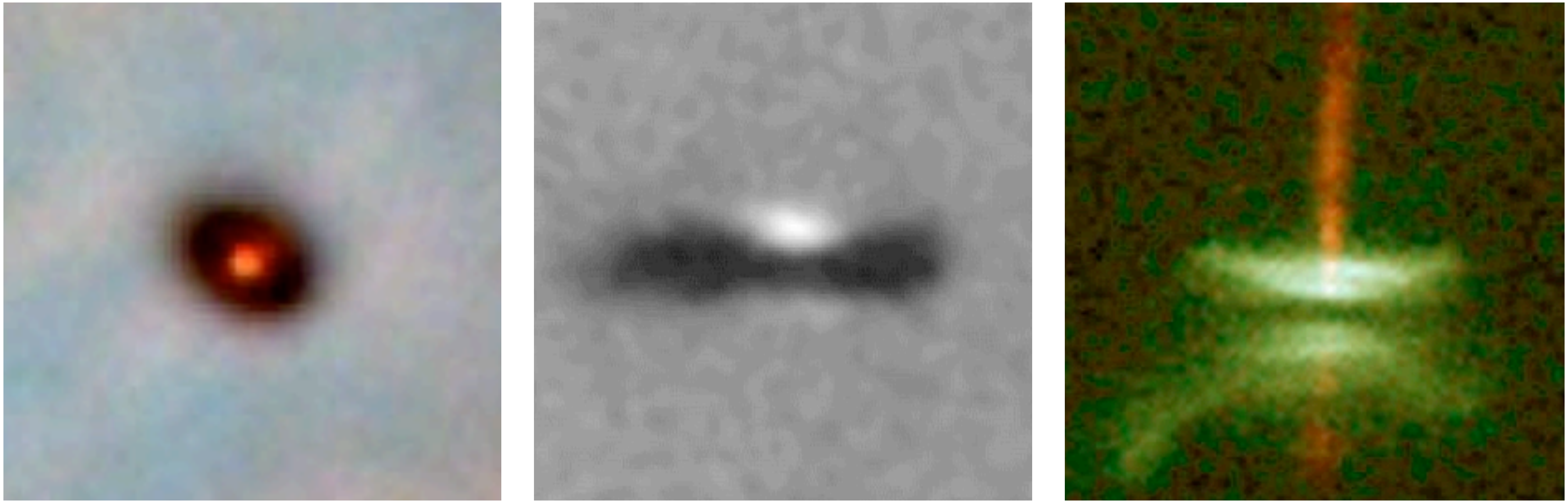
Observations: protoplanetary discs

- Nebular hypothesis (solar nebula):
  Swedenborg, Kant, Laplace (18th century)

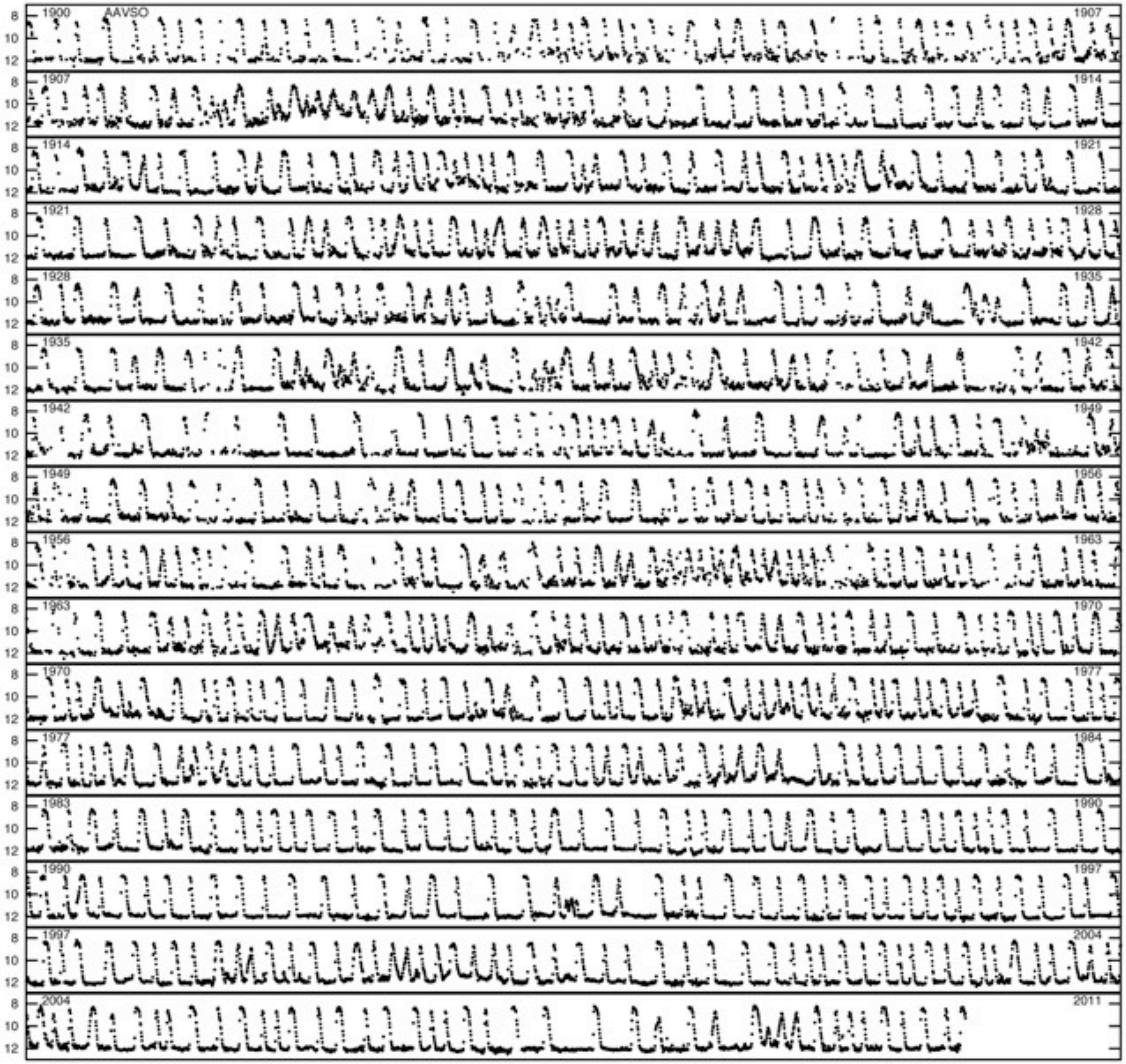
- Direct observations of protoplanetary discs (Hubble ST, 1995-)



- Extrasolar planets around main-sequence stars (1995-)
- Debris discs and transitional discs (Spitzer ST, infrared, 2003-)

Hubble Space Telescope

Observations: cataclysmic variable stars



http://aavso.org/

SS Cygni
dwarf nova

V magnitude
(range 12-8)
1900-2010

`aavso.org`

Also UV and
soft X-rays

FDEPS, Kyoto
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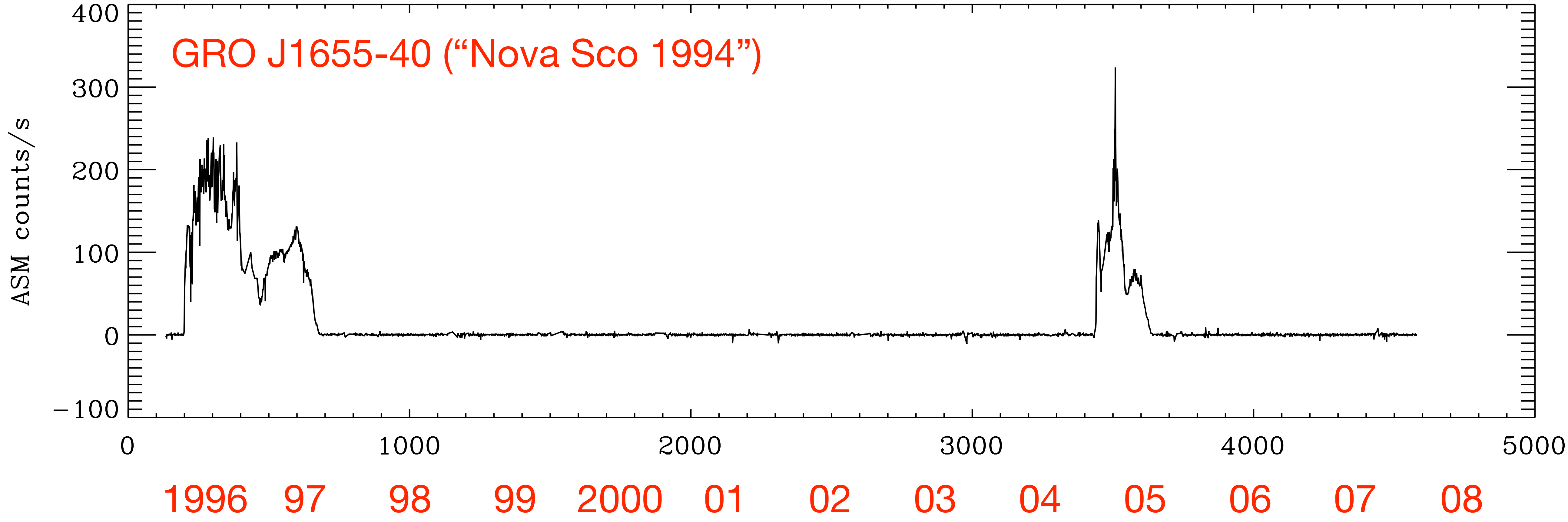
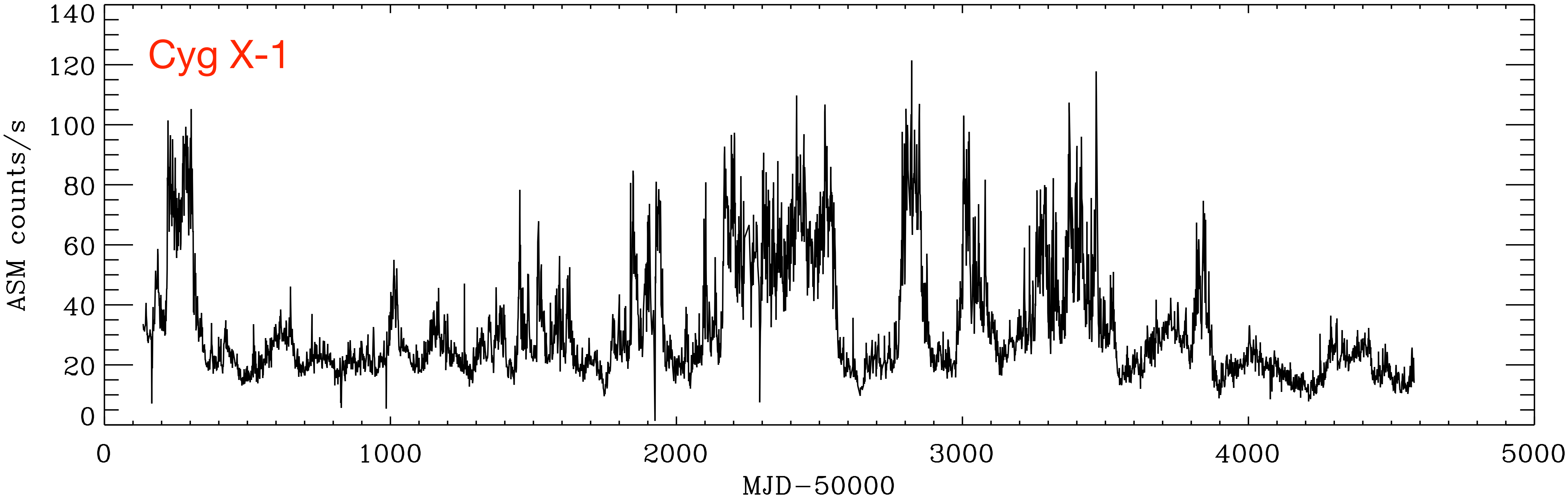
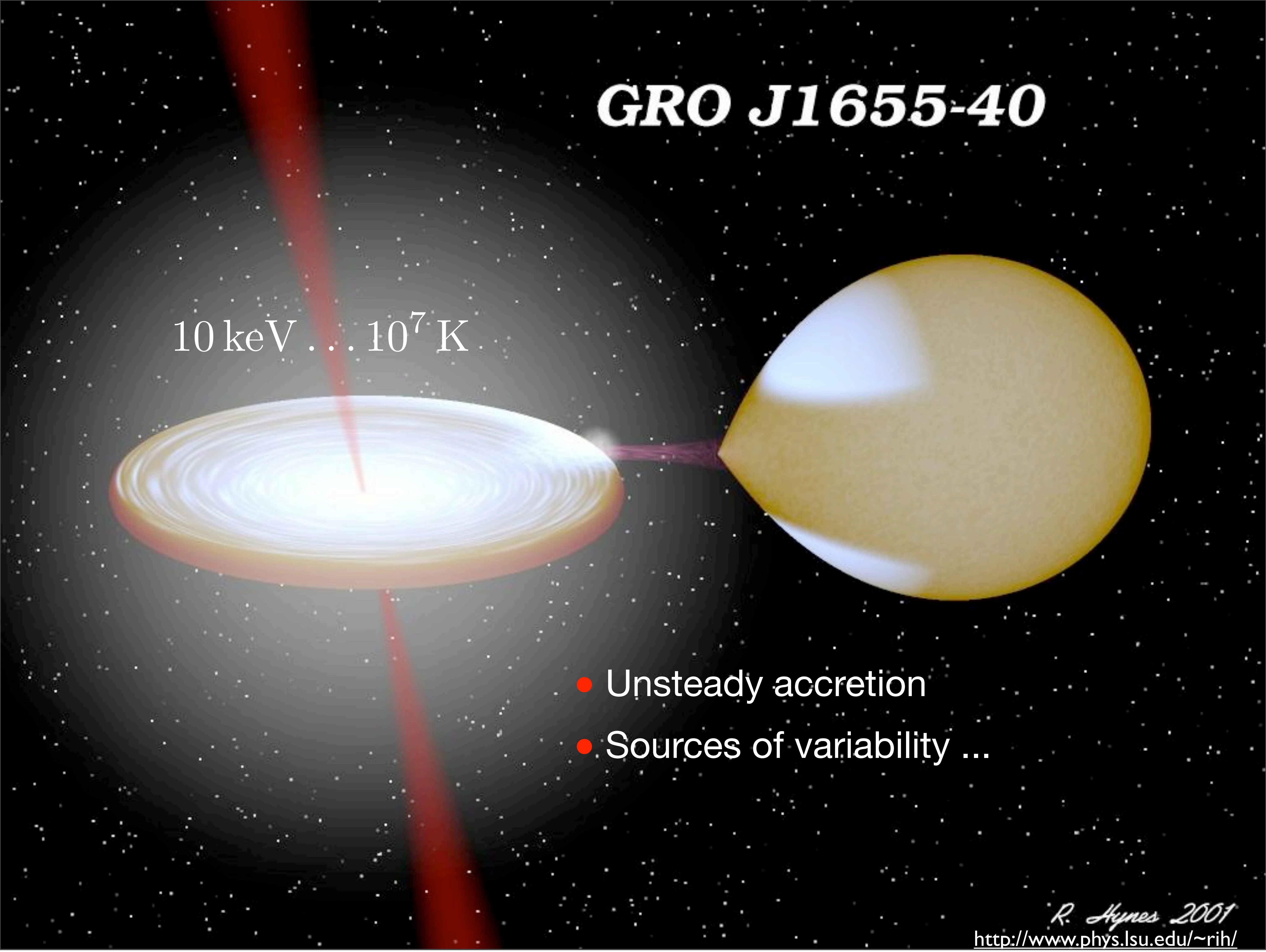
Observations: X-ray binary stars (1960s-)







$10\,\mathrm{keV} \ldots 10^7\,\mathrm{K}$

- Unsteady accretion
- Sources of variability ...

http://www.phys.lsu.edu/~rih/

For an alternative point of view, try http://web.archive.org/web/20111202230603/http://www.accretiondisk.org/

Orbital dynamics

- Test particle in gravitational potential Φ
- Cylindrical polar coordinates (r, ϕ, z)
- Newtonian dynamics

- Assume: $\Phi = \Phi(r, z)$ axisymmetric

  $\Phi(r, -z) = \Phi(r, z)$ symmetric

- Special case: $\Phi = -GM(r^2 + z^2)^{-1/2}$

point-mass potential → Keplerian orbits

Orbital dynamics

- Equation of motion

$$\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla}\Phi \qquad \left\{ \begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -\Phi_{,r} \\ r\ddot{\phi} + 2\dot{r}\dot{\phi} &= 0 \\ \ddot{z} &= -\Phi_{,z} \end{aligned} \right.$$

- Specific energy $\varepsilon = \frac{1}{2}|\dot{\boldsymbol{r}}|^2 + \Phi = \mathrm{const}$
- Specific angular momentum $h = r^2\dot{\phi} = \mathrm{const}$
- Reduces to 2D problem

$$\ddot{r} = -\Phi^{\mathrm{eff}}_{,r}$$

$$\ddot{z} = -\Phi^{\mathrm{eff}}_{,z}$$

- Effective potential $\Phi^{\mathrm{eff}} = \Phi + \dfrac{h^2}{2r^2}$

Orbital dynamics

- Circular orbit in midplane $(z = 0)$

$$0 = \Phi^{\text{eff}}_{,z}(r, 0)$$

✓ by symmetry

$$0 = \Phi^{\text{eff}}_{,r}(r, 0) = \Phi_{,r}(r, 0) - \frac{h^2}{r^3}$$

$$\varepsilon = \frac{h^2}{2r^2} + \Phi(r, 0)$$

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{defining} \quad \begin{matrix} h_\circ(r) \\ \varepsilon_\circ(r) \end{matrix}$$

- Important relation

$$\frac{\mathrm{d}\varepsilon_\circ}{\mathrm{d}r} = \frac{h_\circ}{r^2}\frac{\mathrm{d}h_\circ}{\mathrm{d}r} - \cancel{\frac{h_\circ^2}{r^3}} + \cancel{\Phi_{,r}(r, 0)}$$

$$\frac{\mathrm{d}\varepsilon_\circ}{\mathrm{d}h_\circ} = \frac{h_\circ}{r^2} = \dot{\phi} = \Omega_\circ$$

orbital angular velocity

Orbital dynamics

- Keplerian case

$$\Phi(r,0) = -\frac{GM}{r}$$

$$h_\circ = (GMr)^{1/2}$$

$$\varepsilon_\circ = -\frac{GM}{2r}$$

$$\Omega_\circ = \left(\frac{GM}{r^3}\right)^{1/2}$$

Orbital dynamics

- Reminder of general Keplerian orbits

$$\ddot{\boldsymbol{r}} = -\frac{GM\boldsymbol{r}}{|\boldsymbol{r}|^3}$$

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{r} \times \dot{\boldsymbol{r}}) = \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} = \boldsymbol{0}$$

- Orbit is confined to plane $\perp \boldsymbol{h}$, so introduce polar coordinates (r, ϕ):

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2} \qquad h = r^2\dot{\phi} = \mathrm{const}$$

- Let $r = 1/u$ and note that $\dfrac{\mathrm{d}}{\mathrm{d}t} = \dot{\phi}\dfrac{\mathrm{d}}{\mathrm{d}\phi} = hu^2\dfrac{\mathrm{d}}{\mathrm{d}\phi}$:

$$hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left[hu^2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{u}\right)\right] - h^2u^3 = -GMu^2$$

$$\frac{\mathrm{d}^2u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2}$$

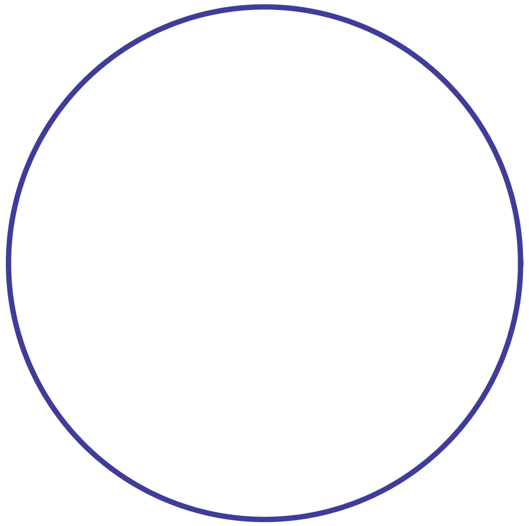
Orbital dynamics

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2}$$

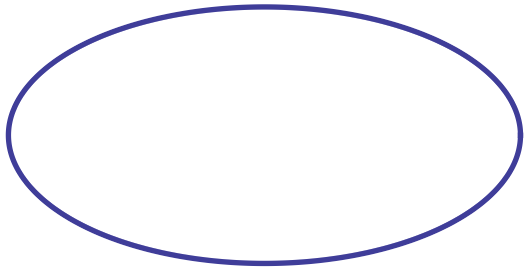
- General solution (with two arbitrary constants)

$$u = \frac{GM}{h^2}[1 + e\cos(\phi - \varpi)] \qquad \Rightarrow \; r = \frac{\lambda}{1 + e\cos(\phi - \varpi)}$$

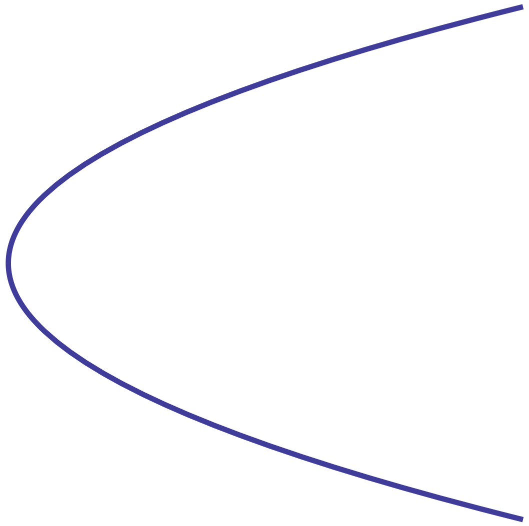
- Polar equation of conic section:



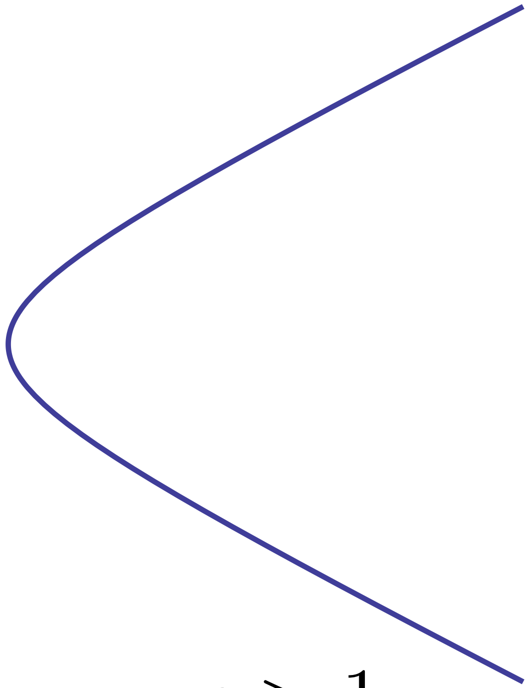
$e = 0$
circle
bound
$(\varepsilon < 0)$



$0 < e < 1$
ellipse
bound



$e = 1$
parabola
marginally
unbound



$e > 1$
hyperbola
unbound
$(\varepsilon > 0)$

Orbital dynamics

- Perturbations $(\delta r, \delta z)$ of circular orbits in midplane (h fixed)

$$\ddot{\delta r} = -\Omega_r^2\,\delta r \qquad \Omega_r^2 = \Phi^{\rm eff}_{,rr}(r,0)$$

$$\ddot{\delta z} = -\Omega_z^2\,\delta z \qquad \Omega_z^2 = \Phi^{\rm eff}_{,zz}(r,0)$$

[$\Phi^{\rm eff}_{,rz}(r,0) = 0$ by symmetry]

Ω_r usually called κ (horizontal) epicyclic frequency

Ω_z sometimes called μ vertical (epicyclic) frequency

- Orbit is stable if $\Omega_r^2 > 0$ (i.e. $\kappa^2 > 0$) and $\Omega_z^2 > 0$
  i.e. if orbit is of minimum energy for given h

Orbital dynamics

- Now

$$\kappa^2 = \Phi_{,rr}(r,0) + \frac{3h_\circ^2}{r^4}$$

$$= \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{h_\circ^2}{r^3}\right) + \frac{3h_\circ^2}{r^4}$$

$$= \frac{1}{r^3}\frac{\mathrm{d}h_\circ^2}{\mathrm{d}r}$$

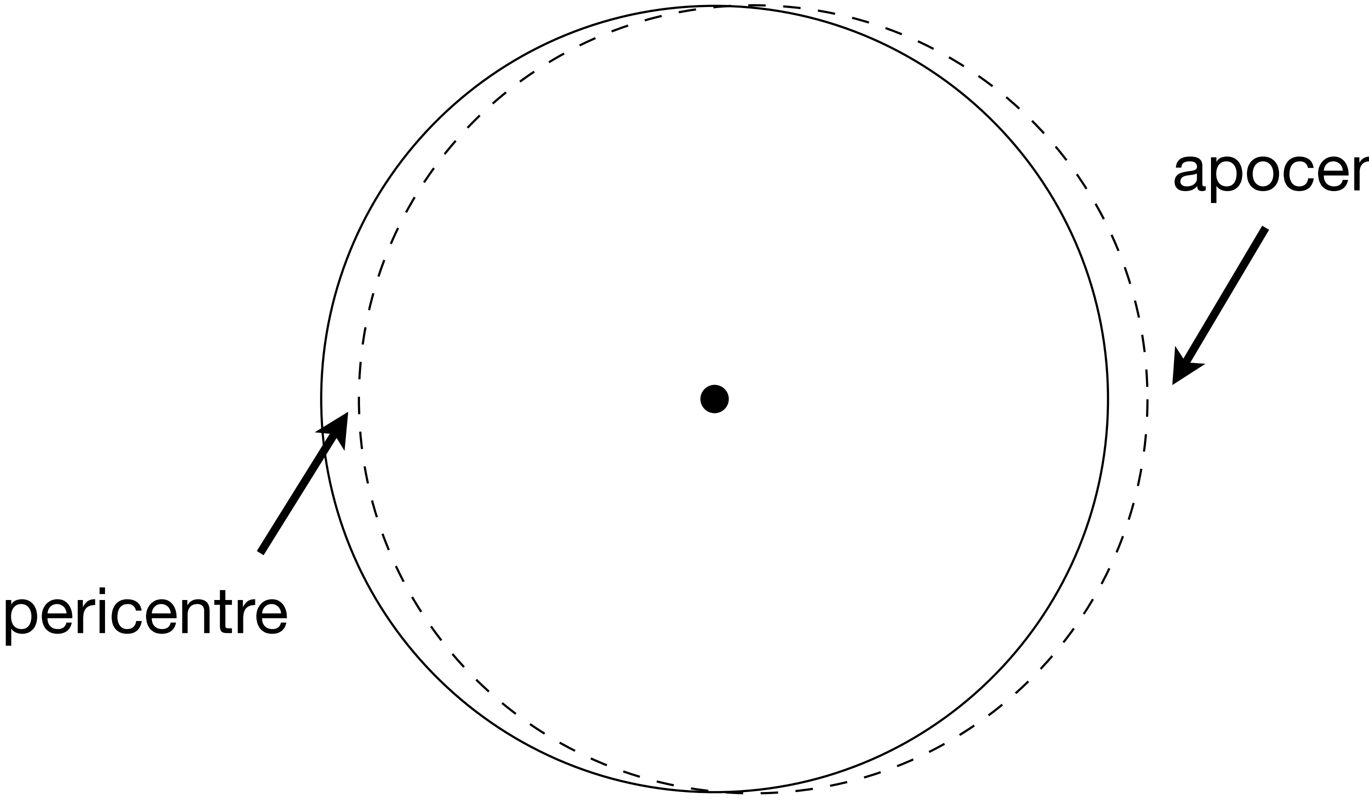
$$= 4\Omega_\circ^2 + 2r\Omega_\circ\frac{\mathrm{d}\Omega_\circ}{\mathrm{d}r}$$

$$\Omega_z^2 = \Phi_{,zz}(r,0)$$

Orbital dynamics

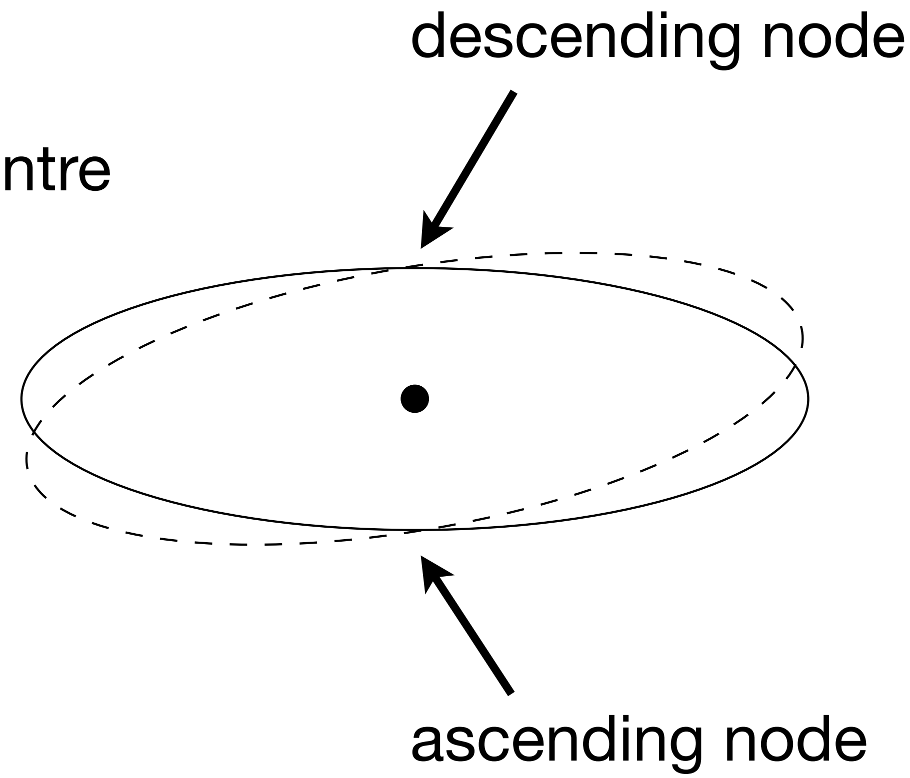
- Keplerian case

$$\kappa = \Omega_z = \Omega$$



$$\kappa = \Omega$$

eccentric Keplerian orbit



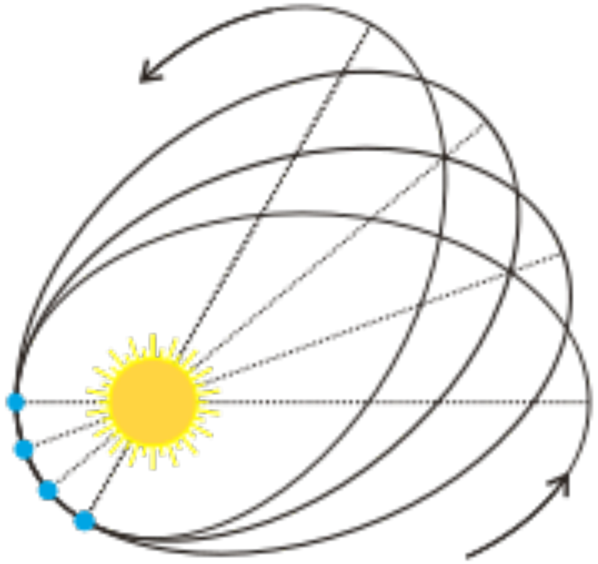
$$\Omega_z = \Omega$$

inclined Keplerian orbit

Orbital dynamics

- Precession
  - If $\kappa \approx \Omega$ and/or $\Omega_z \approx \Omega$, describe as slowly precessing orbit
  - Minimum r (pericentre) occurs at time intervals $\Delta t = \dfrac{2\pi}{\kappa}$

$$\begin{aligned}\Delta\phi &= \frac{2\pi\Omega}{\kappa} \\ &= 2\pi\left(\frac{\Omega}{\kappa} - 1\right) + 2\pi \\ &= 2\pi\left(\frac{\Omega}{\kappa} - 1\right) \mod 2\pi\end{aligned}$$



http://en.wikipedia.org/wiki/File:Perihelion_precession.svg

  - Apsidal precession rate $\dfrac{\Delta\phi}{\Delta t} = \Omega - \kappa$
  - Similarly, nodal precession rate $= \Omega - \Omega_z$

Orbital dynamics

- Example 1: Kerr metric of rotating black hole
- Dimensionless spin parameter: $-1 < a < 1$
- From general relativity: (let $a < 0$ for retrograde orbit)

$$\Omega = \frac{c^3}{GM}\left(\frac{1}{x^{3/2}+a}\right) \qquad x = \frac{r}{GM/c^2}$$

$$\frac{\kappa^2}{\Omega^2} = 1 - 6x^{-1} + 8ax^{-3/2} - 3a^2x^{-2}$$

$$\frac{\Omega_z^2}{\Omega^2} = 1 - 4ax^{-3/2} + 3a^2x^{-2}$$

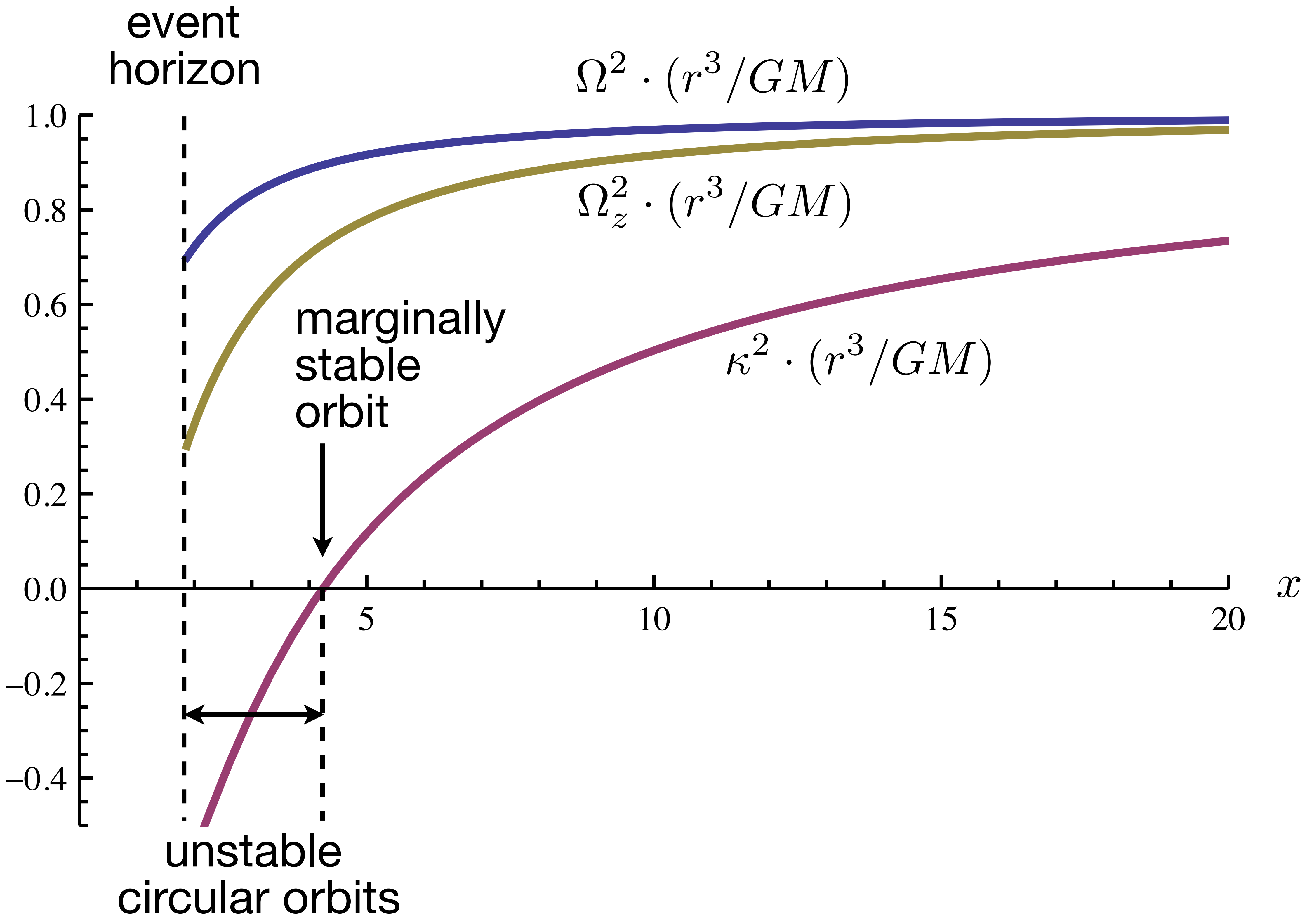
- Precession rates far from black hole ($x \gg 1$):

$$\Omega - \kappa \approx \frac{3c^3}{GMx^{5/2}} = \frac{3(GM)^{3/2}}{c^2r^{5/2}} \qquad \text{Einstein}$$

$$\Omega - \Omega_z \approx \frac{2ac^3}{GMx^3} = \frac{2a(GM)^2}{c^3r^3} \qquad \text{Lense-Thirring}$$

Orbital dynamics

- e.g. $a = 0.5$



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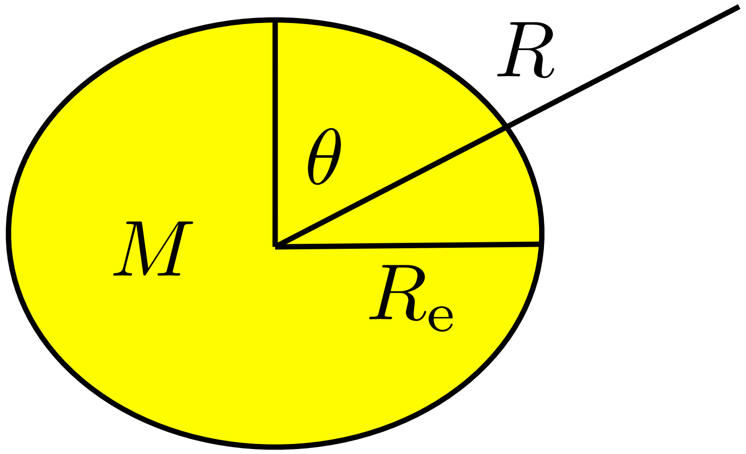
Orbital dynamics

- Example 2: Exterior of rotating planet or star (Newtonian)
- Multipole expansion in spherical polar coordinates (R, θ, ϕ):

$$\Phi = -\frac{GM}{R}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_\mathrm{e}}{R}\right)^n P_n(\cos\theta)\right]$$

multipole coefficient

Legendre polynomial



- e.g. Saturn: $J_2 \approx 1.63 \times 10^{-2}, \; J_4 \approx -9.4 \times 10^{-4}$

Orbital dynamics

$$\Phi = -\frac{GM}{R}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_\mathrm{e}}{R}\right)^n P_n(\cos\theta)\right]$$

- Find:

$$\Omega^2 = \frac{GM}{r^3}\left[1 - \sum_{n=2}^{\infty} (n+1) J_n \left(\frac{R_\mathrm{e}}{r}\right)^n P_n(0)\right]$$

$$\kappa^2 = \frac{GM}{r^3}\left[1 + \sum_{n=2}^{\infty} (n+1)(n-1) J_n \left(\frac{R_\mathrm{e}}{r}\right)^n P_n(0)\right]$$

$$\Omega_z^2 = \frac{GM}{r^3}\left[1 - \sum_{n=2}^{\infty} (n+1)^2 J_n \left(\frac{R_\mathrm{e}}{r}\right)^n P_n(0)\right]$$

- Related by $\kappa^2 + \Omega_z^2 = 2\Omega^2$ (potential satisfies Laplace's equation)

Orbital dynamics

- Precession rates for large r (using $P_2(0) = -1/2$):

$$\Omega - \kappa \approx \frac{3}{2} J_2 \left(\frac{R_\mathrm{e}}{r} \right)^2 \Omega$$

$$\Omega - \Omega_z \approx -\frac{3}{2} J_2 \left(\frac{R_\mathrm{e}}{r} \right)^2 \Omega$$

- e.g. F ring of Saturn:

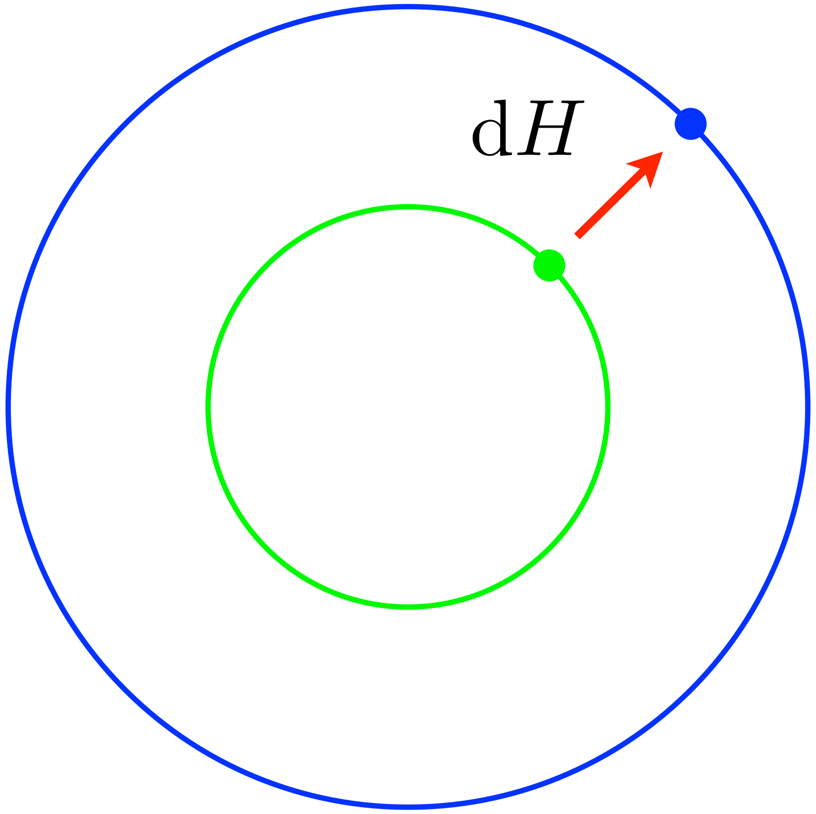
$$\Omega - \kappa \approx 0.0045\,\Omega \approx 2.6^\circ/\mathrm{day}$$

Mechanics of accretion

- Consider two particles in circular orbits in the midplane
- Can energy be released by an exchange of angular momentum?
- Total energy and angular momentum:

$$E = E_1 + E_2 = m_1\varepsilon_1 + m_2\varepsilon_2$$
$$H = H_1 + H_2 = m_1 h_1 + m_2 h_2$$

- In an infinitesimal exchange:



$$\mathrm{d}E = \mathrm{d}E_1 + \mathrm{d}E_2 = m_1\Omega_1\,\mathrm{d}h_1 + m_2\Omega_2\,\mathrm{d}h_2$$
$$\mathrm{d}H = \mathrm{d}H_1 + \mathrm{d}H_2 = m_1\,\mathrm{d}h_1 + m_2\,\mathrm{d}h_2$$

- If $\mathrm{d}H = 0$ then

$$\mathrm{d}E = (\Omega_1 - \Omega_2)\,\mathrm{d}H_1$$

- In practice $\mathrm{d}\Omega/\mathrm{d}r < 0$
- Energy released by transferring angular momentum outwards

Mechanics of accretion

- Generalize argument to allow for exchange of mass:

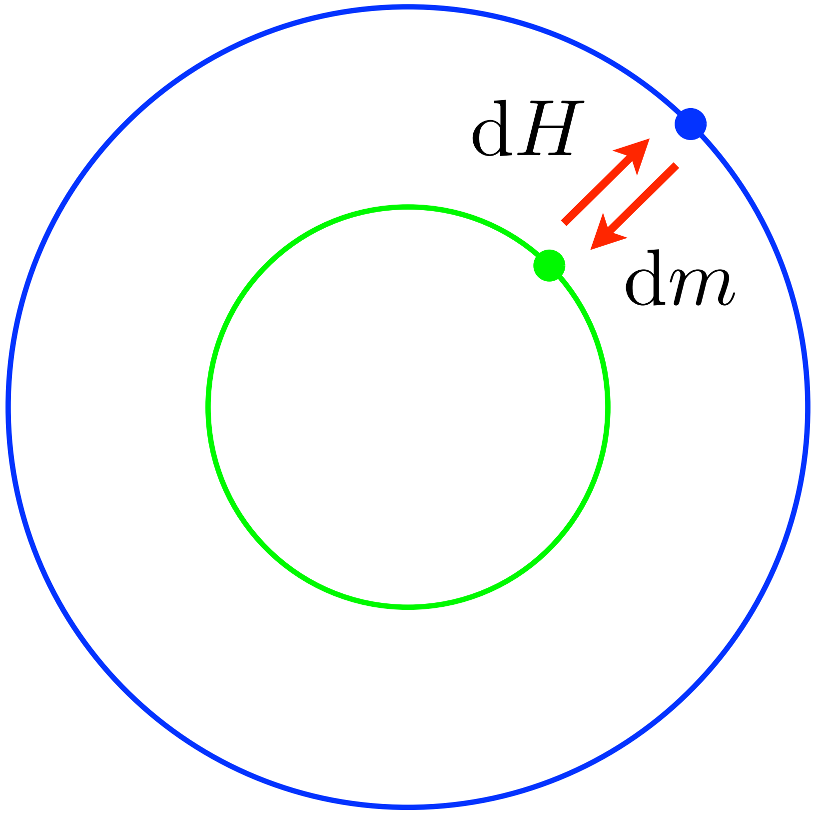
$$\mathrm{d}M = \mathrm{d}m_1 + \mathrm{d}m_2 = 0$$

$$\mathrm{d}H = \mathrm{d}H_1 + \mathrm{d}H_2 = 0 \qquad \mathrm{d}H_i = m_i\,\mathrm{d}h_i + h_i\,\mathrm{d}m_i$$

$$\begin{aligned}\mathrm{d}E_i &= m_i\Omega_i\,\mathrm{d}h_i + \varepsilon_i\,\mathrm{d}m_i \\ &= \Omega_i\,\mathrm{d}H_i + (\varepsilon_i - h_i\Omega_i)\,\mathrm{d}m_i\end{aligned}$$

$$\mathrm{d}E = (\Omega_1 - \Omega_2)\,\mathrm{d}H_1 + [(\varepsilon_1 - h_1\Omega_1) - (\varepsilon_2 - h_2\Omega_2)]\mathrm{d}m_1$$

- In practice $\mathrm{d}(\varepsilon - h\Omega)/\mathrm{d}r = -h\,\mathrm{d}\Omega/\mathrm{d}r > 0$
- Energy released by transferring angular momentum outwards and mass inwards
- This is the physical basis of an accretion disc



Equations of astrophysical fluid dynamics

- Astrophysical fluid dynamics (AFD):
- Basic model: Newtonian gas dynamics:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} \Phi - \frac{1}{\rho} \boldsymbol{\nabla} p$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$\boldsymbol{u}$	velocity
Φ	gravitational potential
ρ	density
p	pressure
γ	adiabatic exponent

  - Compressible
  - Ideal (inviscid, adiabatic)
  - Non-relativistic (Galilean-invariant)
  - Lagrangian (material) derivative:

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}$$

Equations of astrophysical fluid dynamics

- Gravity:
  - Non-self-gravitating fluid:
    - Φ is prescribed (fixed / external potential)
  - Self-gravitating fluid:
    - Φ is determined (in part) from the density of the fluid:

$$\nabla^2\Phi = 4\pi G\rho$$

Equations of astrophysical fluid dynamics

- Extensions of the basic model:
  - Viscosity:
    - Usually extremely small
    - May be needed to provide small-scale dissipation
    - May be introduced to model turbulent transport

$$\frac{\partial \boldsymbol{u}}{\partial t} = \cdots + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\mathbf{T}$$

$$\mathbf{T} = 2\mu\mathbf{S} + \mu_{\mathrm{b}}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}$$

$$\mathbf{S} = \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}}\right] - \frac{1}{3}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}$$

$\mathbf{T}$	viscous stress tensor
μ	(shear) viscosity
μ_{b}	bulk viscosity
$\mathbf{S}$	shear tensor
$\mathbf{I}$	unit tensor

kinematic viscosity $\nu = \mu/\rho$

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Equations of astrophysical fluid dynamics

- Non-adiabatic effects:
  - Thermal energy equation:

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} = \mathcal{H} - \mathcal{C}$$

T	temperature
s	specific entropy
$\mathcal{H}$	heating / unit volume
$\mathcal{C}$	cooling / unit volume (non-adiabatic effects)

  - Heating:
    - Viscous:

$$\mathcal{H} = \mathbf{T} : \boldsymbol{\nabla} \boldsymbol{u} = 2\mu \mathbf{S}^2 + \mu_{\mathrm{b}} (\boldsymbol{\nabla} \cdot \boldsymbol{u})^2$$

  - Cooling:
    - Radiative: $\mathcal{C} = \boldsymbol{\nabla} \cdot \boldsymbol{F}$
    - Diffusion approximation:
      (optically thick regions)

$$\boldsymbol{F} = -\frac{16\sigma T^3}{3\kappa\rho} \boldsymbol{\nabla} T$$

σ	Stefan-Boltzmann constant
κ	opacity (Rosseland mean)

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Equations of astrophysical fluid dynamics

- Equation of state:

$$p = p(\rho, T)$$

  - Ideal gas with radiation:

$$p = p_{\mathrm{g}} + p_{\mathrm{r}} = \frac{k\rho T}{\mu_{\mathrm{m}} m_{\mathrm{p}}} + \frac{4\sigma T^4}{3c}$$

k	Boltzmann constant
μ_{m}	mean molecular weight
m_{p}	proton mass
c	speed of light

  - p_r important at very high T
  - $\mu_{\mathrm{m}} = 0.5$ for fully ionized H, $\mu_{\mathrm{m}} = 2$ for molecular H, etc.

- Thermal energy equation in dynamical variables:

$$\rho T \, \mathrm{d}s = \left(\frac{1}{\gamma_3 - 1}\right)\left(\mathrm{d}p - \frac{\gamma_1 p}{\rho}\mathrm{d}\rho\right)$$

$$\Rightarrow \left(\frac{1}{\gamma_3 - 1}\right)\left(\frac{\mathrm{D}p}{\mathrm{D}t} - \frac{\gamma_1 p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t}\right) = \mathcal{H} - \mathcal{C}$$

- For ideal gas of constant ratio of specific heats, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$

Equations of astrophysical fluid dynamics

- Extensions of the basic model:
  - Magnetohydrodynamics (MHD)
  - Radiation hydrodynamics (RHD)
  - Relativistic formulations
  - Kinetic theory / plasma physics

- Simplifications of the basic model:
  - Incompressible fluid: $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$
  - Boussinesq / anelastic approximations
  - Barotropic fluid: $p = p(\rho)$

Equations of astrophysical fluid dynamics

- Magnetohydrodynamics (MHD):
- Electrically conducting fluid (plasma, metal, weakly ionized gas)
- Pre-Maxwell equations (without displacement current):

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \boldsymbol{E}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

solenoidal constraint

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$$

$\boldsymbol{B}$	magnetic field
$\boldsymbol{E}$	electric field
$\boldsymbol{J}$	electric current density
μ_0	permeability of free space

($\boldsymbol{\nabla} \cdot \boldsymbol{E}$ equation not required)

- Galilean invariance:

$$\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{v}t \qquad \boldsymbol{B}' = \boldsymbol{B}$$

$$t' = t \qquad \boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$$

$$\boldsymbol{J}' = \boldsymbol{J}$$

Equations of astrophysical fluid dynamics

- Ohm's law:

$\boldsymbol{J}' = \sigma \boldsymbol{E}'$ in rest frame of conductor

σ : electrical conductivity

$\Rightarrow \;\; \boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B})$ for conducting fluid with velocity $\boldsymbol{u}(\boldsymbol{x}, t)$

- Combine with Maxwell:

$$\begin{aligned} \frac{\partial \boldsymbol{B}}{\partial t} &= -\boldsymbol{\nabla} \times \boldsymbol{E} \\ &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times \left(\frac{\boldsymbol{J}}{\sigma}\right) \\ &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) - \boldsymbol{\nabla} \times (\eta \boldsymbol{\nabla} \times \boldsymbol{B}) \\ &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \end{aligned}$$

if η uniform

magnetic diffusivity

$$\eta = \frac{1}{\mu_0 \sigma}$$

$\propto$ resistivity

- "Induction equation": vector advection-diffusion equation

cf. vorticity equation $\dfrac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{\omega}) + \nu \nabla^2 \boldsymbol{\omega}$ for $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$

Equations of astrophysical fluid dynamics

- Ideal MHD (perfect conductor: $\sigma \to \infty$, $\eta \to 0$):

$$\begin{aligned}\frac{\partial \boldsymbol{B}}{\partial t} &= \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) \\ &= \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{B} - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \cancel{\boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{B})}\end{aligned}$$

- Magnetic field is “frozen in” to fluid:
  - Field lines behave as material lines
  - Magnetic flux through an open material surface is conserved
- Valid for large magnetic Reynolds number

$$\mathrm{Rm} = \frac{LU}{\eta} \qquad \text{cf.} \quad \mathrm{Re} = \frac{LU}{\nu}$$ (advection versus diffusion)

- Much easier to achieve on astrophysical scales

Equations of astrophysical fluid dynamics

- Lorentz force per unit volume

$$\begin{aligned}\mathbf{F}_{\rm m} &= \boldsymbol{J} \times \boldsymbol{B} \\ &= \frac{1}{\mu_0}(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} \\ &= \frac{1}{\mu_0}\boldsymbol{B} \cdot \boldsymbol{\nabla}\boldsymbol{B} - \boldsymbol{\nabla}\left(\frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\end{aligned}$$

curvature force: magnetic tension

gradient of magnetic pressure

$$T_{\rm m} = \frac{|\boldsymbol{B}|^2}{\mu_0} \qquad p_{\rm m} = \frac{|\boldsymbol{B}|^2}{2\mu_0}$$

(= magnetic energy density)

$$\mathbf{F}_{\rm m} = \boldsymbol{\nabla} \cdot \mathrm{M}$$

$$\mathrm{M} = \frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0} - \frac{|\boldsymbol{B}|^2}{2\mu_0}\mathbf{I}$$

Maxwell stress tensor

If $\boldsymbol{B} = B\,\boldsymbol{e}_z$,

$$\mathrm{M} = \begin{pmatrix} -p_{\rm m} & 0 & 0 \\ 0 & -p_{\rm m} & 0 \\ 0 & 0 & T_{\rm m} - p_{\rm m} \end{pmatrix}$$

Equations of astrophysical fluid dynamics

- Lorentz force:
  - Magnetic tension + frozen-in field → Alfvén waves

$$v_\mathrm{a} = \left(\frac{T_\mathrm{m}}{\rho}\right)^{1/2}$$ cf. elastic string

$$\boldsymbol{v}_\mathrm{a} = (\mu_0 \rho)^{-1/2} \boldsymbol{B}$$ vector Alfvén velocity

  - Magnetic pressure → magnetoacoustic waves

Equations of astrophysical fluid dynamics

- Ideal MHD equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} \Phi - \frac{1}{\rho} \boldsymbol{\nabla} p + \frac{1}{\mu_0 \rho} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho = -\rho \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} p = -\gamma p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B})$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$$

- Or can expand out $\times\times$
- $\boldsymbol{E}$ and $\boldsymbol{J}$ eliminated
- Nonlinearities in equation of motion and induction equation

Equations of astrophysical fluid dynamics

- Total energy equation in ideal MHD:

$$\frac{\partial}{\partial t}\left[\rho(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+\underset{\substack{\uparrow\\ \text{specific}\\ \text{internal}\\ \text{energy}}}{e})+\frac{|\boldsymbol{B}|^2}{2\mu_0}\right]+\boldsymbol{\nabla}\cdot\left[\rho\boldsymbol{u}(\tfrac{1}{2}|\boldsymbol{u}|^2+\Phi+\underset{\substack{\uparrow\\ \text{specific}\\ \text{enthalpy}}}{w})+\underset{\substack{\uparrow\\ \text{Poynting}\\ \text{flux}}}{\frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}}\right]=0$$

$$\boldsymbol{E}=-\boldsymbol{u}\times\boldsymbol{B}$$

- For ideal gas of constant γ :

$$e=\frac{p}{(\gamma-1)\rho}$$

$$w=e+\frac{p}{\rho}=\frac{\gamma p}{(\gamma-1)\rho}$$

- With self-gravity, $\Phi=\Phi_{\rm int}+\Phi_{\rm ext}$ and only $\frac{1}{2}\Phi_{\rm int}+\Phi_{\rm ext}$ contributes to the energy density

Equations of astrophysical fluid dynamics

- Forces as the divergences of a stress tensor:
- Equation of motion can be written

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\rho\boldsymbol{\nabla}\Phi+\boldsymbol{\nabla}\cdot\mathbf{T}$$

- Related to conservative form for momentum:

$$\frac{\partial}{\partial t}(\rho\boldsymbol{u})+\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}\boldsymbol{u}-\mathbf{T})=-\rho\boldsymbol{\nabla}\Phi$$

- Contributions to stress tensor $\mathbf{T}$:
  - pressure $-p\,\mathbf{I}$
  - viscous $2\mu\mathbf{S}+\mu_{\mathrm{b}}(\boldsymbol{\nabla}\cdot\boldsymbol{u})\mathbf{I}$
  - magnetic $\dfrac{\boldsymbol{B}\boldsymbol{B}}{\mu_0}-\dfrac{|\boldsymbol{B}|^2}{2\mu_0}\mathbf{I}$
  - self-gravity $-\dfrac{\boldsymbol{g}\boldsymbol{g}}{4\pi G}+\dfrac{|\boldsymbol{g}|^2}{8\pi G}\mathbf{I}$ (check using Poisson's equation)

$$\nabla^2\Phi=4\pi G\rho \qquad \boldsymbol{g}=-\boldsymbol{\nabla}\Phi$$

  - also turbulent stresses from correlations of fluctuating fields

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